Applications of Derivatives

Case Study Based Questions

Case Study 1

The fuel cost per hour for running a train is proportional to the square of the speed it generates in km per hour. If the fuel cost ₹ 48 per hour at speed 16 km/h and the fixed charges to run the train amount to ₹ 1200 per hour.

Assume the speed of the train as v km/h.

Based on the above information, solve the following questions.

(CBSE SQP 2021 Term-1)

Q 1. Given that the fuel cost per hour is *k* times the square of the speed the train generates in km/h, the value of *k* is:

a. $\frac{16}{3}$ b. $\frac{1}{3}$ c. 3 d. $\frac{3}{16}$

Q 2. If the train has travelled a distance of 500 km, then the total cost of running the train is given by function:

a. $\frac{15}{16}v + \frac{600000}{v}$ b. $\frac{375}{4}v + \frac{600000}{v}$

c. $\frac{5}{16}v^2 + \frac{150000}{v}$ d. $\frac{3}{16}v + \frac{6000}{v}$

Q 3. The most economical speed to run the train is:

a. 18 km/h b. 5 km/h

c. 80 km/h d. 40 km/h

Q 4. The fuel cost for the train to travel 500 km at the most economical speed is:

a. ₹ 3750
b. ₹ 750
c. ₹ 7500
d. ₹ 75000

Q 5. The total cost of the train to travel 500 km at the most economical speed is:

a. ₹ 3750
b. ₹ 75000
c. ₹ 7500
d. ₹ 15000

Solutions

1. Given, fuel cost = ₹ 48/hour

and speed = 16 km/hour.

According to question, the fuel cost per hour is k times the square of the speed the train generates in km/h.

i.e., Fuel cost $= k\,(\text{speed})^2$

$$\Rightarrow \quad 48 = k\,(16)^2$$

$$\Rightarrow \quad k = \frac{48}{16 \times 16} = \frac{3}{16}$$

So, option (d) is correct.

2. Total cost of running train (let C)

$$= \text{Fuel cost} + \text{Fixed charge} = \frac{3}{16} v^2 t + 1200t$$

Distance covered = 500 km

$$\Rightarrow \quad \text{time} = \frac{500}{v} \text{ hrs}$$

Total cost of running train 500 km

$$= \frac{3}{16} v^2 \left(\frac{500}{v}\right) + 1200 \left(\frac{500}{v}\right)$$

$$\Rightarrow \quad C = \frac{375}{4} v + \frac{600000}{v}$$

So, option (b) is correct.

3. We have, $C = \dfrac{375}{4} v + \dfrac{600000}{v}$

Differentiate w.r.t. v, we get

$$\frac{dC}{dv} = \frac{375}{4} - \frac{600000}{v^2}$$

For economical speed of train,

$$\frac{dC}{dv} = \frac{375}{4} - \frac{600000}{v^2} = 0$$

$$\Rightarrow \quad v^2 = \frac{600000 \times 4}{375}$$

$$\Rightarrow \quad v^2 = 6400 \text{ km/h}$$

$$\Rightarrow \quad v = 80 \text{ km/h}$$

So, option (c) is correct.

4. Fuel cost for the train to travel 500 km

$$= \frac{375}{4} v$$

$\therefore$ Fuel cost for most economical speed $= \frac{375}{4} \times 80$

$$= ₹\,7500$$

So, option (c) is correct.

5. Total cost of the train to travel 500 km

$$= \frac{375}{4} v + \frac{600000}{v}$$

$\therefore$ Total cost for most economical speed

$$= \frac{375}{4} \times 80 + \frac{600000}{80}$$

$$= 7500 + 7500 = ₹\,15000$$

So, option (d) is correct.

Case Study 2

In a residential society comprising of 100 houses, there were 60 children between the ages of 10-15 years. They were inspired by their teachers to start composting to ensure that biodegradable waste is recycled. For this purpose, instead of each child doing it for only his/her house, children convinced the Residents welfare association to do it as a society initiative. For this, they identified a square area in the local park. Local authorities charged amount of ₹ 50 per square metre for space so that there is

no misuse of the space and Resident welfare association takes it seriously.
Association hired a labourer for digging out 250 m^3 and he charged ₹ 400 × $(\text{depth})^2$.

Association will like to have minimum cost.

Based on the given information, solve the following questions.

(CBSE SQP 2021 Term-1)

Q 1. Let side of square plot is x metre and its depth is h metre, then cost c for the pit is:

a. $\frac{50}{h} + 400\,h^2$ b. $\frac{12500}{h} + 400\,h^2$

c. $\frac{250}{h} + h^2$ d. $\frac{250}{h} + 400\,h^2$

Q 2. Value of h (in m) for which $\frac{dc}{dh} = 0$ is:

a. 1.5 b. 2 c. 2.5 d. 3

Q 3. $\frac{d^2c}{dh^2}$ is given by:

a. $\frac{25000}{h^3} + 800$ b. $\frac{500}{h^3} + 800$

c. $\frac{100}{h^3} + 800$ d. $\frac{500}{h^3} + 2$

Q 4. Value of x (in m) for minimum cost is:

a. 5 b. $10\sqrt{\frac{5}{3}}$ c. $5\sqrt{5}$ d. 10

Q 5. Total minimum cost of digging the pit (in ₹) is:

a. 4100 b. 7500 c. 7850 d. 3220

Solutions

1. Let each side of square shaped pit be x metre and depth as h metre.

As volume of earth taken out $= 250\,\text{m}^3$

$$\Rightarrow \quad x \cdot x \cdot h = 250\,\text{m}^3$$

$$\Rightarrow \quad x^2 = \frac{250}{h} \quad \ldots(1)$$

Since, local authorities charged amount of ₹ 50 per square metre for space.

$$\therefore \quad \text{Cost of land} = ₹\,50 \times \text{space for land}$$

$$= ₹\,50 \times x^2$$

$$= ₹\,50 \times \frac{250}{h} = ₹\,\frac{12500}{h} \text{ [from eq. (1)]}$$

So, total cost = cost of land + cost of labour

$$\Rightarrow \quad c = \frac{12500}{h} + 400 \times (\text{depth})^2$$

$$\Rightarrow \quad c = \frac{12500}{h} + 400\,h^2 \quad \ldots(2)$$

So option (b) is correct.

2. Now, $\dfrac{dc}{dh} = \dfrac{d}{dh}\left\{\dfrac{12500}{h} + 400\,h^2\right\}$

$$= -\frac{12500}{h^2} + 800\,h \quad \ldots(3)$$

$$\because \quad \frac{dc}{dh} = 0 \quad \text{[given]}$$

$$\Rightarrow \quad \frac{-12500}{h^2} + 800\,h = 0$$

$$\Rightarrow \quad 800\,h^3 = 12500 \quad [\because h \neq 0]$$

$$\Rightarrow \quad h^3 = \frac{125}{8} = \left(\frac{5}{2}\right)^3$$

$$\Rightarrow \quad h = 2.5\,\text{m}$$

So, option (c) is correct.

3. Differentiate eq. (3) w.r.t. h, we get

$$\frac{d^2c}{dh^2} = \frac{d}{dh}\left\{-\frac{12500}{h^2} + 800\,h\right\}$$

$$= \frac{25000}{h^3} + 800$$

So, option (a) is correct.

4. Now, value of x (in m) for minimum cost

$$= \sqrt{\frac{250}{h}} \qquad \text{[from eq. (1)]}$$

$$\left[\because \left.\frac{d^2c}{dh^2}\right|_{\text{at } h = 2.5} > 0\right]$$

$$= \sqrt{\frac{250}{2.5}} \qquad [\because h = 2.5]$$

$$= \sqrt{100} = 10$$

So, option (d) is correct.

5. Total minimum cost of digging the pit (in ₹)

$$= \frac{12500}{h} + 400\,h^2 \qquad \text{[from eq. (2)]}$$

$$= \frac{12500}{2.5} + 400\,(2.5)^2 \qquad [\because h = 2.5]$$

$$= 5000 + 2500 = ₹\ 7500$$

So, option (b) is correct.

Case Study 3

Let f be continuous on $[a, b]$ and differentiable on the open interval (a, b) then:

(a) f is strictly increasing in $[a, b]$ if $f'(x) > 0$ for each $x \in (a, b)$.

(b) f is strictly decreasing in $[a, b]$ if $f'(x) < 0$ for each $x \in (a, b)$.

(c) f is a constant function in $[a, b]$ if $f'(x) = 0$ for each $x \in (a, b)$.

Based on the above information, solve the following questions:

Q 1. The function $f(x) = \cos(x)$ is strictly increasing in:

a. $(\pi, 2\pi)$ b. $(0, \pi)$ c. $\left(\frac{\pi}{2}, \pi\right)$ d. $(0, 2\pi)$

Q 2. The function $f(x) = 3x + 17$ is strictly increasing in:

a. R_{-} b. $R_{\oplus}$ c. R d. $Z_{\oplus}$

Q 3. The function $f(x) = \sin(x)$ is:

a. strictly increasing in $\left(0, \frac{\pi}{2}\right)$

b. strictly decreasing in $\left(0, \frac{\pi}{2}\right)$

c. strictly increasing in $\left(\frac{\pi}{2}, \pi\right)$

d. None of the above

Q 4. The function $f(x) = e^{2x}$ is strictly increasing on:

a. only $Z_{\oplus}$

b. only $R_{\oplus}$

c. R

d. only R_{-}

Q 5. The function $f(x) = \log(\sin(x))$ is strictly increasing on:

Solutions

1. Given, $f(x) = \cos x$

Then, $f'(x) = -\sin x$

In interval $(\pi, 2\pi)$,

$$f'(x) > 0 \qquad [\because \sin x < 0]$$

Therefore, $f(x)$ is strictly increasing on $(\pi, 2\pi)$.

So, option (a) is correct.

2. Given, $f(x) = 3x + 17$

Differentiate w.r.t. x, we get

$f'(x) = 3 > 0$, in every interval of R.

Thus, the function is strictly increasing on R.

So, option (c) is correct.

3. Given function is $f(x) = \sin x$.

Differentiate w.r.t. x, we get

$$f'(x) = \cos x$$

Since, for each $x \in \left(0, \frac{\pi}{2}\right)$, $\cos x > 0$; therefore we have

$f'(x) > 0$ $\qquad [\because \cos x$ is positive in first quadrant$]$

Hence, f is strictly increasing in $\left(0, \frac{\pi}{2}\right)$.

So, option (a) is correct.

4. Given, $f(x) = e^{2x}$

Differentiate w.r.t. x, we get

$f'(x) = 2e^{2x} > 0$ in every interval of R.

Thus, the function is strictly increasing on R.

So, option (c) is correct.

5. Given, $f(x) = \log \sin x$

Then, $f'(x) = \dfrac{1}{\sin x} \times \cos x = \cot x$

In interval $\left(0, \dfrac{\pi}{2}\right)$,

$$f'(x) > 0$$

Therefore, $f(x)$ is strictly increasing on $\left(0, \dfrac{\pi}{2}\right)$.

So, option (a) is correct.

Case Study 4

An architecture design an auditorium for a school for its cultural activities. The floor of the auditorium is rectangular in shape and has a fixed perimeter P.

Based on the above information, solve the following questions:

Q 1. If x and y represents the length and breadth of the rectangular region, then relation between the variable is:

a. $x + y = P$

b. $x^2 + y^2 = P^2$

c. $2(x + y) = P$

d. $x + 2y = P$

Q 2. The area (A) of the rectangular region, as a function of x, can be expressed as:

a. $A = Px + \dfrac{x}{2}$

b. $A = \dfrac{Px + x^2}{2}$

c. $A = \dfrac{Px - 2x^2}{2}$

d. $A = \dfrac{x^2}{2} + Px^2$

Q 3. School's manager is interested in maximising the area of floor 'A' for this to be happen, the value of x should be:

a. P b. $\frac{P}{2}$ c. $\frac{P}{3}$ d. $\frac{P}{4}$

Q 4. The value of y, for which the area of floor is maximum, is:

a. $\frac{P}{2}$ b. $\frac{P}{3}$ c. $\frac{P}{4}$ d. $\frac{P}{16}$

Q 5. Maximum area of floor is:

a. $\frac{P^2}{16}$ b. $\frac{P^2}{64}$ c. $\frac{P^2}{4}$ d. $\frac{P^2}{28}$

Solutions

1. Perimeter of floor = 2 (length + breadth)

$\Rightarrow \quad P = 2(x+y)$

So, option (c) is correct.

2. Area, A = length × breadth

$\Rightarrow \quad A = xy \quad \ldots(1)$

Since, $\quad P = 2(x+y)$

$\Rightarrow \quad \frac{P-2x}{2} = y$

From eq. (1), $\quad A = xy$

$\Rightarrow \quad A = x\left(\frac{P-2x}{2}\right)$

$\Rightarrow \quad A = \frac{Px-2x^2}{2}$

So, option (c) is correct.

3. We have, $\quad A = \frac{1}{2}(Px-2x^2)$

$\Rightarrow \quad \frac{dA}{dx} = \frac{1}{2}(P-4x) = 0$

For maximum or minimum of A,

$\Rightarrow \quad \frac{dA}{dx} = 0$

$$\Rightarrow \qquad P-4x=0 \quad \Rightarrow \quad x=\frac{P}{4}$$

Clearly, at $x=\frac{P}{4}, \frac{d^2A}{dx^2}=-2<0$

$\therefore$ Area is maximum at $x=\frac{P}{4}$.

So, option (d) is correct.

4. We have, $y=\frac{P-2x}{2}=\frac{P}{2}-\frac{P}{4}=\frac{P}{4}$ $\left[\because x=\frac{P}{4}\right]$

So, option (c) is correct.

5. We have, $A=xy=\frac{P}{4}\cdot\frac{P}{4}=\frac{P^2}{16}$ (maximum area)

So, option (a) is correct.

Case Study 5

The relation between the height of the plant ('y' in cm) with respect to its exposure to the sunlight is governed by the following equation $y=4x-\frac{1}{2}x^2$, where 'x' is the number of days exposed to the sunlight, for $x \le 3$.

Based on the above information, solve the following questions: (CBSE SQP 2023-24)

Q1. Find the rate of growth of the plant with respect to the number of days exposed to the sunlight.

Q2. Does the rate of growth of the plant increase or decrease in the first three days? What will be the height of the plant after 2 days?

Solutions

1. Given equation is $y = 4x - \frac{1}{2}x^2$.

The rate of growth of the plant with respect to the number of days exposed is

$$\frac{dy}{dx} = \left(4 - \frac{2x}{2}\right) = 4 - x$$

2. $\because \frac{dy}{dx} = 4 - x$

At $x = 1, 2, 3$; $\frac{dy}{dx} > 0$

Hence, the rate of growth of the plant increase in first three days.

$\because \quad y = 4x - \frac{1}{2}x^2$

At $x = 2$,

$$y = 4 \times 2 - \frac{1}{2}(2)^2$$
$$= 8 - 2 = 6$$

Case Study 6

The shape of a toy is given as $f(x) = 8x^4 - 4x^2 + 3$

To make the toy beautiful 2 sticks which are perpendicular to each other were placed at a point (4, 5) above the toy.

Based on the above information, solve the following questions:

Q 1. Find the abscissa of the critical point of the function $f(x)$.

Q 2. Find the maximum value of the function.

Q 3. At which of the following intervals will $f(x)$ be decreasing?

Solutions

1. For the critical point of $f(x)$, $f'(x) = 0$

$$\Rightarrow \quad \frac{d}{dx}\{8x^4 - 4x^2 + 3\} = 0$$

$$\Rightarrow \quad 32x^3 - 8x = 0$$

$$\Rightarrow \quad 8x(4x^2 - 1) = 0$$

$$\Rightarrow \quad x = 0 \text{ or } 4x^2 - 1 = 0$$

$$\Rightarrow \quad x = 0 \text{ or } x^2 = \frac{1}{4}$$

$$\Rightarrow \quad x = 0 \text{ or } x = \pm\frac{1}{2}$$

Thus, abscissa of critical points are 0 and $\pm\frac{1}{2}$.

2. $\because \quad f'(x) = 32x^3 - 8x$

$$\therefore \quad f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}(32x^3 - 8x) = 96x^2 - 8$$

At $x = \pm\frac{1}{2}$,

$$f''\left(\pm\frac{1}{2}\right) = 96\left(\pm\frac{1}{2}\right)^2 - 8$$

$$= 96 \times \frac{1}{4} - 8 = 24 - 8 = 16 > 0$$

At $x = 0$,

$f''(0) = 96(0)^2 - 8 = 0 - 8 = -8 < 0$

So, the function is maximum at $x = 0$ and minimum at $x = \pm\frac{1}{2}$.

$\therefore$ Maximum value of function is

$$f(0) = 8(0)^4 - 4(0)^2 + 3 = 0 - 0 + 3 = 3.$$

3. $\because \quad f(x) = 8x^4 - 4x^2 + 3$

$$\therefore \quad f'(x) = 32x^3 - 8x$$

Putting, $f'(x) = 0$, we get,

$$32x^3 - 8x = 0$$

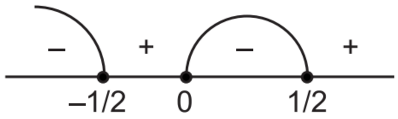
$$\Rightarrow \quad 8x(4x^2 - 1) = 0$$

$$\Rightarrow \quad 8x(2x-1)(2x+1)=0$$

$$\Rightarrow \quad x=0,\ -\frac{1}{2} \text{ and } \frac{1}{2}$$

which divides real line into four intervals

$$\left(-\infty, \frac{-1}{2}\right), \left(\frac{-1}{2}, 0\right), \left(0, \frac{1}{2}\right) \text{ and } \left(\frac{1}{2}, \infty\right).$$



Therefore, $f(x)$ is decreasing in $\left(-\infty, \frac{-1}{2}\right)$ and $\left(0, \frac{1}{2}\right)$

i.e., $$\left(-\infty, \frac{-1}{2}\right) \cup \left(0, \frac{1}{2}\right)$$

Case Study 7

P(x) = $-5x^2 + 125x + 37500$ is the total profit function of a company, where x is the production of the company.

Based on the above information, solve the following questions:

Q1. What will be the production when the profit is maximum?

Q2. What will be the maximum profit?

Or

When the production is 2 units, what will be the profit of the company?

Q3. Find the interval in which the profit function is strictly increasing.

Solutions

1. Given, $$P(x) = -5x^2 + 125x + 37{,}500$$
$$P'(x) = -10x + 125$$
$$P''(x) = -10$$

For maximum or minimum, we have

$$P'(x) = 0$$

$$\Rightarrow \quad -10x + 125 = 0$$

$$\Rightarrow \quad x = 12.5$$

Clearly, $P''(x) < 0$ for $x = 12.5$

Thus, profit is maximum when $x = 12.5$.

2. Maximum profit is given by,

$$P(12.5) = -5(12.5)^2 + 125(12.5) + 37{,}500$$
$$= -781.25 + 1562.5 + 37{,}500$$
$$= ₹\ 38{,}281.25$$

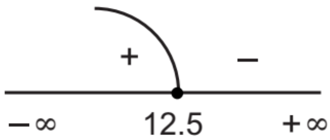
Or

$$P(2) = -5(2)^2 + 125(2) + 37500$$
$$= -20 + 250 + 37500 = ₹\ 37730$$

3. Given, profit function $P(x) = -5x^2 + 125x + 37500$

Differentiate w.r.t. x, we get

$$P'(x) = -10x + 125x$$
$$= -10(x - 12.5)$$



Since, for each $x \in (-\infty, 12.5)$

$$P'(x) > 0$$

Hence, $P(x)$ *i.e.*, profit function is strictly increasing in $(-\infty, 12.5)$.

Case Study 8

In an elliptical sport field, the authority wants to design a rectangular soccer field with the maximum possible area. The sport field is given by the graph of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Based on the above information, solve the following questions: (CBSE SQP 2022-23)

Q1. If the length and the breadth of the rectangular field are 2x and 2y respectively, then find the area function in terms of x.

Q2. Find the critical point of the function.

Q3. Use first derivative test to find the length 2x and width 2y of the soccer field (in terms of a and b) that maximise its area.

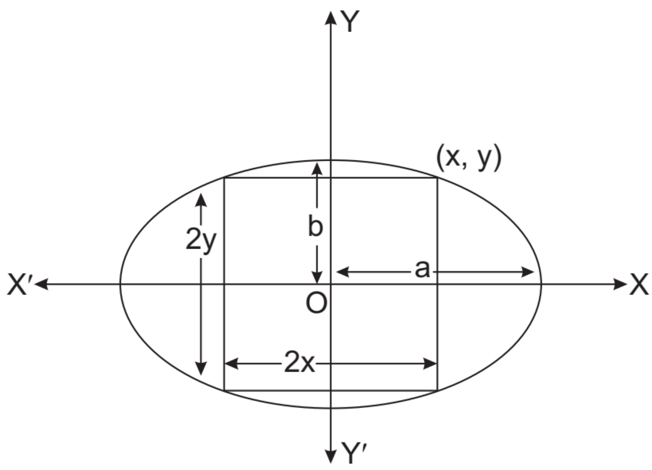
Or

Use Second Derivative Test to find the length 2x and width 2y of the soccer field (in terms of a and b) that maximise its area.

Solutions

1. Given equation of elliptical sport field is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \ldots(1)$$



Here, a = length of semi-major axis

and b = length of semi-minor axis

From eq. (1),

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\Rightarrow \quad y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad \text{[for first quadrant]}$$

Let $(x, y) = \left(x, \frac{b}{a}\sqrt{a^2 - x^2}\right)$ be the upper right vertex of the rectangle.

$\therefore$ The area function A = length $\times$ breadth

$$= 2x \times 2y = 4x\frac{b}{a}\sqrt{a^2 - x^2}$$

$$= \frac{4b}{a}x\sqrt{a^2 - x^2}, x \in (0, a) \qquad \text{...(2)}$$

2. Now differentiating eq. (2) w.r.t. x, we get

$$\frac{dA}{dx} = \frac{4b}{a}\left[\sqrt{a^2 - x^2} + x \times \frac{-2x}{\sqrt{a^2 - x^2}} \times \frac{1}{2}\right]$$

$$= \frac{4b}{a}\left\{\frac{a^2 - x^2 - x^2}{\sqrt{a^2 - x^2}}\right\}$$

$$= \frac{-4b}{a} \times \frac{2\left(x^2 - \frac{a^2}{2}\right)}{\sqrt{a^2 - x^2}} = \frac{-8b}{a} \times \frac{\left(x + \frac{a}{\sqrt{2}}\right)\left(x - \frac{a}{\sqrt{2}}\right)}{\sqrt{a^2 - x^2}}$$

For maximum or minimum of A,

$$\frac{dA}{dx} = 0$$

$$\Rightarrow \quad -\frac{8b}{a} \times \frac{\left(x + \frac{a}{\sqrt{2}}\right)\left(x - \frac{a}{\sqrt{2}}\right)}{\sqrt{a^2 - x^2}} = 0$$

$$\Rightarrow \quad x = \frac{a}{\sqrt{2}} \qquad \left[\because x \neq \pm a, -\frac{a}{\sqrt{2}} \text{ i.e., can't be negative}\right]$$

So, $x = \frac{a}{\sqrt{2}}$ is the critical point.

3. For the values of x less than $\frac{a}{\sqrt{2}}$ and close to $\frac{a}{\sqrt{2}}$,

$$\frac{dA}{dx} > 0$$

and for the values of x greater than $\frac{a}{\sqrt{2}}$ and close to $\frac{a}{\sqrt{2}}$,

$$\frac{dA}{dx} < 0$$

Hence, by using the first derivative test, there is a local maximum at the critical point $x = \frac{a}{\sqrt{2}}$. Since, there is only one critical point, therefore the area of the soccer field is maximum at this critical point $x = \frac{a}{\sqrt{2}}$.

Thus, for maximum area of the soccer field,

length of the soccer field $= 2x = 2 \times \frac{a}{\sqrt{2}} = a\sqrt{2}$

and width of the soccer field $= 2y = \frac{2b}{a}\sqrt{a^2 - x^2}$

$$= \frac{2b}{a}\sqrt{a^2 - \frac{a^2}{2}} = \frac{2b}{a} \times \frac{a}{\sqrt{2}} = b\sqrt{2}.$$

Or

From part (1), $A = \frac{4b}{a} x\sqrt{a^2 - x^2}, x \in (0, a)$

Squaring on both sides, we get

$$z = A^2 = \frac{16b^2}{a^2} x^2(a^2 - x^2) = \frac{16b^2}{a^2}(a^2x^2 - x^4), x \in (0, a)$$

Here, A is maximum as z is maximum.

Now, differentiate w.r.t. x, we get

$$\frac{dz}{dx} = \frac{16b^2}{a^2}(2a^2x - 4x^3) = \frac{32b^2}{a^2} \cdot x\,(a^2 - 2x^2)$$

$$= \frac{32b^2}{a^2} \cdot x\,(a + \sqrt{2}x)\,(a - \sqrt{2}x)$$

For maximum or minimum of z, $\frac{dz}{dx} = 0$

$$\therefore \quad \frac{32b^2}{a^2} . x\,(a+\sqrt{2}x)\,(a-\sqrt{2}x) = 0$$

$$\Rightarrow \quad x = \frac{a}{\sqrt{2}}$$

$$[\because x \text{ can't be zero or negative, } \therefore\ x \neq 0, -\frac{a}{\sqrt{2}}]$$

Again differentiate w.r.t. x, we get

$$\frac{d^2z}{dx^2} = \frac{32b^2}{a^2}\{a^2 - 6x^2\}$$

$$\therefore \text{ At } x = \frac{a}{\sqrt{2}}, \frac{d^2z}{dx^2} = \frac{32b^2}{a^2}\left(a^2 - 6 \times \frac{a^2}{2}\right)$$

$$= \frac{32b^2}{a^2} \times (-2a^2) = -64b^2 < 0$$

Hence, by using second derivative test, there is a local maximum value of z at the critical point $x = \frac{a}{\sqrt{2}}$. Since, there is only one critical point, therefore z is maximum at $x = \frac{a}{\sqrt{2}}$, hence A is maximum at $x = \frac{a}{\sqrt{2}}$.

Thus, for maximum area of the soccer field,

length of the soccer field $= 2x = 2 \times \frac{a}{\sqrt{2}} = a\sqrt{2}$

and width of the soccer field $= 2y = \frac{2b}{a}\sqrt{a^2 - x^2}$

$$= \frac{2b}{a}\sqrt{a^2 - \frac{a^2}{2}}$$

$$= \frac{2b}{a} \times \frac{a}{\sqrt{2}} = b\sqrt{2}$$

Case Study 9

The temperature of a person during an intestinal illness is given by $f(x) = -0.1x^2 + mx + 98.6$, $0 \le x \le 12$, m being a constant, where $f(x)$ is the temperature in °F at x days.

Based on the above information, solve the following questions: (CBSE SQP 2022-23)

Q1. Is the function differentiable in the interval (0, 12) ? Justify your answer.

Q2. If 6 is the critical point of the function, then find the value of the constant m.

Q3. Find the intervals in which the function is strictly increasing/strictly decreasing.

Or

Find the points of local maximum/local minimum, if any, in the interval (0, 12) as well as the points of absolute maximum/absolute minimum in the interval [0, 12]. Also, find the corresponding local maximum/local minimum and the absolute maximum/absolute minimum values of the function.

Solutions

1. Given, function $f(x) = -0.1x^2 + mx + 98.6$

Here, $f(x)$ being a polynomial function. So, it is differentiable everywhere.

Hence, function $f(x)$ is differentiable in the interval (0, 12).

2. Now, $f'(x) = -0.2x + m$

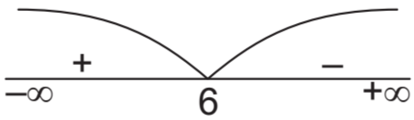
Since, $x = 6$ is the critical point of the function.

$\therefore \quad f'(6) = 0 \quad \Rightarrow \quad -0.2 \times 6 + m = 0 \quad \Rightarrow \quad m = 1.2$

3. Given, $f(x) = -0.1x^2 + mx + 98.6,\ x \in [0, 12]$

$\Rightarrow \quad f'(x) = -0.2x + m = -0.2x + 1.2 \qquad [\because m = 1.2]$

$= -0.2\,(x - 6)$



Intervals	Sign of $f'(x)$	Conclusion
$(-\infty, 6)$	+	f is strictly increasing in (0, 6)
(6, 12)	–	f is strictly decreasing in (6, 12)

Hence, $f(x)$ is strictly increasing in (0, 6) and strictly decreasing in (6, 12) because $x \in [0, 12]$.

Or

Given, $f(x) = -0.1x^2 + mx + 98.6,\ x \in [0, 12]$

$\Rightarrow \quad f(x) = -0.1x^2 + 1.2x + 98.6 \qquad [\because m = 1.2]$

Differentiate both sides w.r.t. 'x', we get

$f'(x) = -0.2x + 1.2 = 0.2\,(-x + 6)$

For maximum or minimum of $f(x)$,

$f'(x) = 0.2\,(-x + 6) = 0$

$\Rightarrow \quad x = 6$

Again differentiate both sides w.r.t. 'x', we get

$f''(x) = -0.2$

At $x = 6$, $\quad f''(6) = -0.2 < 0$

Hence, by second derivative test, $f(x)$ is maximum at $x = 6$. So, $x = 6$ is a point of local maximum in (0, 12).

Now, local maximum value $= f(6)$

$= -0.1(6)^2 + 1.2\,(6) + 98.6$

$= -0.1 \times 36 + 1.2 \times 6 + 98.6$

$$= -3.6 + 7.2 + 98.6 = 102.2$$

At $x = 0$, $\quad f(0) = -0.1 \times 0 + 1.2 \times 0 + 98.6$

$$= 98.6$$

At $x = 6$, $\quad f(6) = 102.2$

At $x = 12$, $\quad f(12) = -0.1 \times (12)^2 + 1.2 \times 12 + 98.6$

$$= -14.4 + 14.4 + 98.6 = 98.6$$

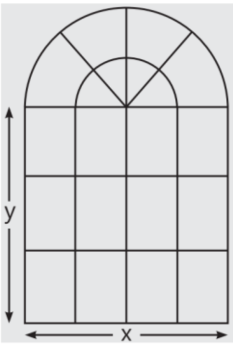
So, $x = 6$ is the point of absolute maximum and $x = 0$, 12 are the points of absolute minimum in the interval [0, 12].

Now, absolute maximum value $= f(6) = 102.2$

and absolute minimum value $= f(0)$ or $f(12) = 98.6$.

Case Study 10

Rohan, a student of class XII, visited his uncles's flat with his father. He observes that the window of the house is in the form of a rectangle surmounted by a semicircular opening having perimeter 10 m as shown in the figure.



Based on the given information, solve the following questions:

Q1. If x and y represent the length and breadth of the rectangular region, then find the area (A) of the window in terms of x.

Q2. Rohan is interested in maximising the area of the whole window, for this to happen, find the value of x.

Q3. Find the maximum area of the window.

Or

For maximum value of A, find the breadth of rectangular part of the window.

Solutions

1. Given, perimeter of window = 10 m

$\therefore \quad x + y + y +$ circumference of semi-circle $= 10$

$$\Rightarrow \qquad x + 2y + \pi\frac{x}{2} = 10 \qquad \ldots(1)$$

So, area of window (A) = Area of rectangle + Area of semi-circle

$$\Rightarrow \qquad A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$$

$$= x\left(5 - \frac{x}{2} - \frac{\pi x}{4}\right) + \frac{1}{2}\cdot\frac{\pi x^2}{4}$$

$$\left[\because \text{From eq. (1), } y = 5 - \frac{x}{2} - \frac{\pi x}{4}\right]$$

$$= 5x - \frac{x^2}{2} - \frac{\pi x^2}{4} + \frac{\pi x^2}{8} = 5x - \frac{x^2}{2} - \frac{\pi x^2}{8}$$

2. We have, $A = 5x - \frac{x^2}{2} - \frac{\pi x^2}{8}$

Differentiate w.r.t. x, we get

$$\frac{dA}{dx} = 5 - x - \frac{\pi x}{4}$$

For maximum or minimum of A,

$$\frac{dA}{dx} = 0 \quad \Rightarrow \quad 5 = x + \frac{\pi x}{4}$$

$$\Rightarrow \qquad x(4+\pi) = 20 \quad \Rightarrow \quad x = \frac{20}{4+\pi}$$

Clearly, $\frac{d^2A}{dx^2} < 0$ at $x = \frac{20}{4+\pi}$

Hence, required value of x is $\frac{20}{4+\pi}$ m.

3. At $x = \frac{20}{4+\pi}$,

$$A = 5\left(\frac{20}{4+\pi}\right) - \left(\frac{20}{4+\pi}\right)^2\frac{1}{2} - \frac{\pi}{8}\left(\frac{20}{4+\pi}\right)^2$$

$$= \frac{100}{4+\pi} - \frac{200}{(4+\pi)^2} - \frac{50\pi}{(4+\pi)^2}$$

$$= \frac{(4+\pi)(100) - 200 - 50\pi}{(4+\pi)^2}$$

$$= \frac{400 + 100\pi - 200 - 50\pi}{(4+\pi)^2}$$

$$= \frac{200 + 50\pi}{(4+\pi)^2} = \frac{50(4+\pi)}{(4+\pi)^2} = \frac{50}{4+\pi} m^2$$

Or

Since, area 'A' is maximum at $= \frac{20}{4+\pi}$.

Now, $y = 5 - \frac{x}{2} - \frac{\pi x}{4} = 5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)$

$$= 5 - x\left(\frac{2+\pi}{4}\right) = 5 - \left(\frac{20}{4+\pi}\right)\left(\frac{2+\pi}{4}\right)$$

$$= 5 - 5\frac{(2+\pi)}{4+\pi} = \frac{20 + 5\pi - 10 - 5\pi}{4+\pi}$$

$$= \frac{10}{4+\pi} m$$

Case Study 11

An open water tank of aluminium sheet of negligible thickness, with a square base and vertical sides, is to be constructed in a farm for irrigation. It should hold 32000 L of water, that comes out from a tube well.

Based on the above information, solve the following questions:

Q1. If the length, width and height of the open tank be x, x and y m respectively, then find the outer surface area of tank in terms of x.

Q2. Show that the cost of material will be least when width of tank is equal to twice of its depth.

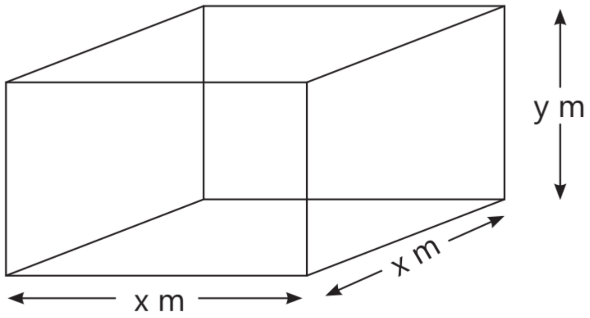
Q3. If cost of aluminium sheet is ₹ 360/m², then find the minimum cost for the construction of tank.

Solutions

1. Since, volume of tank should be 32000 L.

$\therefore \quad x^2y \text{ m}^3 = 32000 \text{ L} = 32 \text{ m}^3 \qquad [\because 1 \text{ litre} = 0.001 \text{ m}^3]$

So, $\qquad x^2y = 32 \qquad \ldots(1)$



Since, the tank is open from the top, therefore the surface area

$$= [x \times x + 2(xy + yx)]$$
$$= [x^2 + 2(2xy)]$$
$$= (x^2 + 4xy) \text{ m}^2$$

Let S be the outer surface area of tank.

Then, $\quad S = x^2 + 4xy$

$\Rightarrow \quad S(x) = x^2 + 4x \cdot \dfrac{32}{x^2} = x^2 + \dfrac{128}{x} \quad [\because x^2y = 32] \ldots(2)$

2. Differentiate eq. (2) w.r.t. x on both sides, we get

$$\frac{dS}{dx} = 2x - \frac{128}{x^2}$$

and $\quad \dfrac{d^2S}{dx^2} = 2 + \dfrac{256}{x^3}$

For maximum or minimum values of S, put $\dfrac{dS}{dx} = 0$

$\therefore \quad 2x = \dfrac{128}{x^2}$

$\Rightarrow \quad x^3 = 64$

$\Rightarrow \quad x = 4 \text{ m}$

At $x = 4$, $\quad \dfrac{d^2S}{dx^2} = 2 + \dfrac{256}{4^3}$

$$= 2 + 4 = 6 > 0$$

$\therefore$ S is minimum when $x = 4$

Now as $x^2y = 32$, therefore $y = 2$

Thus, $x = 2y$

Since, surface area is minimum when $x = 2y$, therefore cost of material will be least when $x = 2y$.

Thus, cost of material will be least when width is equal to twice of its depth. **Hence proved.**

3. Since, minimum surface area

$$= x^2 + 4xy = 4^2 + 4 \times 4 \times 2 = 48\ m^2$$

and cost per m^2 = ₹ 360

$\therefore$ Minimum cost = ₹ (48×360) = ₹ 17280

Case Study 12

Sooraj's father wants to construct a rectangular garden using a brick wall on one side of the garden and wire fencing for the other three sides as shown in the figure. He has 200 metres of fencing wire.

Based on the above information, solve the following questions: (CBSE 2023)

Q1. Let 'x' metre denotes the length of the side of the garden perpendicular to the brick wall and 'y' metre denotes the length of the side parallel to the brick wall. Determine the relation representing the total length of fencing wire and also write A(x), the area of the garden.

Q2. Determine the maximum value of A(x).

Solutions

1. Given, x and y are the length and breadth of the rectangular garden and length of wire is 200 m. The relation representing the given problem is

$$2x + y = 200$$

Now, area of the garden, $A(x) = xy$

$$= x(200 - 2x)$$

$$= (200x - 2x^2)\ m^2$$

2. $\because \; A(x) = 200x - 2x^2$

Differentiate w.r.t. x, we get

$$A'(x) = 200 - 4x$$

For maxima and minima, put $A'(x) = 0$

$\therefore \quad 200 - 4x = 0 \;\Rightarrow\; x = 50$

Now, $A''(x) = -4 < 0 \; \forall \; x$, so A is maximum at $x = 50$.

Hence, maximum value of $A(x)$ is

$$A(50) = 200 \times 50 - 2(50)^2$$
$$= 10000 - 5000 = 5000 \text{ m}^2$$

Solutions for Questions 13 to 32 are Given Below

Case Study 13

Megha wants to prepare a handmade gift box for her friend's birthday at home. For making lower part of box, she takes a square piece of cardboard of side 20 cm.

Based on the above information, answer the following questions.

(i) If x cm be the length of each side of the square cardboard which is to be cut off from corners of the square piece of side 20 cm, then possible value of x will be given by the interval

(a) [0, 20] (b) (0, 10) (c) (0, 3) (d) None of these

(ii) Volume of the open box formed by folding up the cutting corner can be expressed as

(a) $V = x(20 - 2x)(20 - 2x)$ (b) $V = \frac{x}{2}(20 + x)(20 - x)$

(c) $V = \frac{x}{3}(20 - 2x)(20 + 2x)$ (d) $V = x(20 - 2x)(20 - x)$

(iii) The values of x for which $\frac{dV}{dx} = 0$, are

(a) 3, 4 (b) $0, \frac{10}{3}$ (c) 0, 10 (d) $10, \frac{10}{3}$

(iv) Megha is interested in maximising the volume of the box. So, what should be the side of the square to be cut off so that the volume of the box is maximum?

(a) 12 cm (b) 8 cm (c) $\frac{10}{3}$ cm (d) 2 cm

(v) The maximum value of the volume is

(a) $\frac{17000}{27}$ cm^3 (b) $\frac{11000}{27}$ cm^3

(c) $\frac{8000}{27}$ cm^3 (d) $\frac{16000}{27}$ cm^3

Case Study 14

Shobhit's father wants to construct a rectangular garden using a brick wall on one side of the garden and wire fencing for the other three sides as shown in figure. He has 200 ft of wire fencing.

Based on the above information, answer the following questions.

(i) To construct a garden using 200 ft of fencing, we need to maximise its

(a) volume (b) area (c) perimeter (d) length of the side

(ii) If x denote the length of side of garden perpendicular to brick wall and y denote the length of side parallel to brick wall, then find the relation representing total amount of fencing wire.

(a) $x + 2y = 150$ (b) $x + 2y = 50$ (c) $y + 2x = 200$ (d) $y + 2x = 100$

(iii) Area of the garden as a function of x, say $A(x)$, can be represented as

(a) $200 + 2x^2$ (b) $x - 2x^2$ (c) $200x - 2x^2$ (d) $200 - x^2$

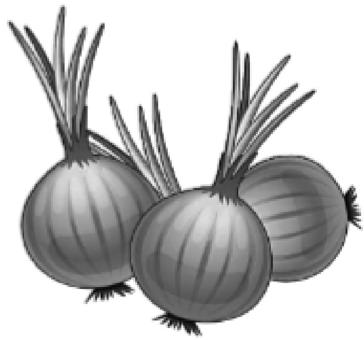
(iv) Maximum value of $A(x)$ occurs at x equals

(a) 50 ft (b) 30 ft (c) 26 ft (d) 31 ft

(v) Maximum area of garden will be

(a) 2500 sq. ft (b) 4000 sq. ft (c) 5000 sq. ft (d) 6000 sq. ft

Case Study 15



The Government declare that farmers can get ₹ 300 per quintal for their onions on 1st July and after that,the price will be dropped by ₹ 3 per quintal per extra day. Shyam's father has 80 quintal of onions in the field on 1st July and he estimates that crop is increasing at the rate of 1 quintal per day.

Based on the above information, answer the following questions.

(i) If x is the number of days after 1st July, then price and quantity of onion respectively can be expressed as

(a) ₹ $(300 - 3x)$, $(80 + x)$ quintals (b) ₹ $(300 - 3x)$, $(80 - x)$ quintals

(c) ₹ $(300 + x)$, 80 quintals (d) None of these

(ii) Revenue R as a function of x can be represented as

(a) $R(x) = 3x^2 - 60x - 24000$ (b) $R(x) = -3x^2 + 60x + 24000$

(c) $R(x) = 3x^2 + 40x - 16000$ (d) $R(x) = 3x^2 - 60x - 14000$

(iii) Find the number of days after 1st July, when Shyam's father attain maximum revenue.

(a) 10 (b) 20 (c) 12 (d) 22

(iv) On which day should Shyam's father harvest the onions to maximise his revenue?

(a) 11th July (b) 20th July (c) 12th July (d) 22nd July

(v) Maximum revenue is equal to

(a) ₹ 20,000 (b) ₹ 24,000 (c) ₹ 24,300 (d) ₹ 24,700

Case Study 16

An owner of an electric bike rental company have determined that if they charge customers ₹ x per day to rent a bike, where $50 \le x \le 200$, then number of bikes (n), they rent per day can be shown by linear function $n(x) = 2000 - 10x$. If they charge ₹ 50 per day or less, they will rent all their bikes. If they charge ₹ 200 or more per day, they will not rent any bike.

Based on the above information, answer the following questions.

(i) Total revenue R as a function of x can be represented as

(a) $2000x - 10x^2$ (b) $2000x + 10x^2$

(c) $2000 - 10x$ (d) $2000 - 5x^2$

(ii) If $R(x)$ denote the revenue, then maximum value of $R(x)$ occur when x equals

(a) 10 (b) 100 (c) 1000 (d) 50

(iii) At $x = 260$, the revenue collected by the company is

(a) ₹ 10 (b) ₹ 500 (c) ₹ 0 (d) ₹ 1000

(iv) The number of bikes rented per day, if $x = 105$ is

(a) 850 (b) 900 (c) 950 (d) 1000

(v) Maximum revenue collected by company is

(a) ₹ 40,000 (b) ₹ 50,000 (c) ₹ 75,000 (d) ₹ 1,00,000

Case Study 17

Mr. Sahil is the owner of a high rise residential society having 50 apartments. When he set rent at ₹ 10000/month, all apartments are rented. If he increases rent by ₹ 250/ month, one fewer apartment is rented. The maintenance cost for each occupied unit is ₹ 500/month.

Based on the above information answer the following questions.

(i) If P is the rent price per apartment and N is the number of rented apartment, then profit is given by

(a) NP (b) $(N - 500)P$ (c) $N(P - 500)$ (d) none of these

(ii) If x represent the number of apartments which are not rented, then the profit expressed as a function of x is

(a) $(50 - x)(38 + x)$ (b) $(50 + x)(38 - x)$ (c) $250(50 - x)(38 + x)$ (d) $250(50 + x)(38 - x)$

(iii) If $P = 10500$, then $N =$

(a) 47 (b) 48 (c) 49 (d) 50

(iv) If $P = 11{,}000$, then the profit is

(a) ₹ 4,83,000 (b) ₹ 5,00,000 (c) ₹ 5,05,000 (d) ₹ 6,50,000

(v) The rent that maximizes the total amount of profit is

(a) ₹ 11000 (b) ₹ 11500 (c) ₹ 15800 (d) ₹ 16500

Case Study 18

Western music concert is organised every year in the stadium that can hold 36000 spectators. With ticket price of ₹ 10, the average attendance has been 24000. Some financial expert estimated that price of a ticket should be determined by the function $p(x) = 15 - \frac{x}{3000}$, where x is the number of tickets sold.

Based on the above information, answer the following questions.

(i) The revenue, R as a function of x can be represented as

(a) $15x - \frac{x^2}{3000}$ (b) $15 - \frac{x^2}{3000}$ (c) $15x - \frac{1}{30000}$ (d) $15x - \frac{x}{3000}$

(ii) The range of x is

(a) [24000, 36000] (b) [0, 24000] (c) [0, 36000] (d) none of these

(iii) The value of x for which revenue is maximum, is

(a) 20000 (b) 21000 (c) 22500 (d) 25000

(iv) When the revenue is maximum, the price of the ticket is

(a) ₹ 5 (b) ₹ 5.5 (c) ₹ 7 (d) ₹ 7.5

(v) How many spectators should be present to maximize the revenue?

(a) 21500 (b) 21000 (c) 22000 (d) 22500

Case Study 19

A tin can manufacturer designs a cylindrical tin can for a company making sanitizer and disinfector. The tin can is made to hold 3 litres of sanitizer or disinfector.

Based on the above in formation, answer the following questions.

(i) If r cm be the radius and h cm be the height of the cylindrical tin can, then the surface area expressed as a function of r as

(a) $2\pi r^2$ (b) $2\pi r^2 + 6000$ (c) $2\pi r^2 + \frac{5000}{r}$ (d) $2\pi r^2 + \frac{6000}{r}$

(ii) The radius that will minimize the cost of the material to manufacture the tin can is

(a) $\sqrt[3]{\frac{600}{\pi}}$ cm (b) $\sqrt{\frac{500}{\pi}}$ cm (c) $\sqrt[3]{\frac{1500}{\pi}}$ cm (d) $\sqrt{\frac{1500}{\pi}}$ cm

(iii) The height that will minimize the cost of the material to manufacture the tin can is

(a) $\sqrt[3]{\frac{1500}{\pi}}$ cm (b) $2\sqrt[3]{\frac{1500}{\pi}}$ cm (c) $\sqrt{\frac{1500}{\pi}}$ (d) $2\sqrt{\frac{1500}{\pi}}$

(iv) If the cost of material used to manufacture the tin can is ₹ 100/m^2 and $\sqrt[3]{\frac{1500}{\pi}} \approx 7.8$, then minimum cost is approximately

(a) ₹ 11.538 (b) ₹ 12 (c) ₹ 13 (d) ₹ 14

(v) To minimize the cost of the material used to manufacture the tin can, we need to minimize the

(a) volume (b) curved surface area

(c) total surface area (d) surface area of the base

Case Study 20

A poster is to be formed for a company advertisement. The top and bottom margins of poster should be 9 cm and the side margins should be 6 cm. Also, the area for printing the advertisement should be 864 cm^2.

Based on the above information, answer the following questions.

(i) If a cm be the width and b cm be the height of poster, then the area of poster, expressed in terms of a and b, is given by

(a) $648 + 18a + 12b$ (b) $18a + 12b$

(c) $584 + 18a + 12b$ (d) none of these

(ii) The relation between a and b is given by

(a) $a = \frac{648+12b}{b-18}$ (b) $a = \frac{12b}{b-18}$ (c) $a = \frac{12b}{b+18}$ (d) none of these

(iii) Area of poster in terms of b is given by

(a) $\frac{12b^2}{b-18}$ (b) $\frac{648b+12b^2}{b-18}$ (c) $\frac{648b+12b^2}{b+18}$ (d) $\frac{12b^2}{b+18}$

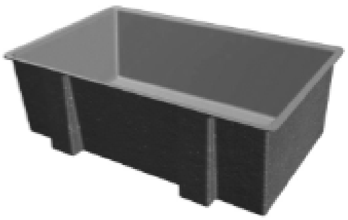
(iv) The value of b, so that area of the poster is minimized, is

(a) 54 (b) 36 (c) 27 (d) 22

(v) The value of a, so that area of the poster is minimized, is

(a) 24 (b) 36 (c) 40 (d) 22

Case Study 21



Nitin wants to construct a rectangular plastic tank for his house that can hold 80 ft^3 of water. The top of the tank is open. The width of tank will be 5 ft but the length and heights are variables. Building the tank cost ₹ 20 per sq. foot for the base and ₹ 10 per sq. foot for the side.

Based on the above information, answer the following questions.

(i) In order to make a least expensive water tank, Nitin need to minimize its

(a) Volume (b) Base (c) Curved surface area (d) Cost

(ii) Total cost of tank as a function of h can be represented as

(a) $c(h) = 100\,h - 320 - 1600/h$ (b) $c(h) = 100\,h - 320\,h - 720\,h^2$

(c) $c(h) = 100 + 320\,h + 1600\,h^2$ (d) $c(h) = 100\,h + 320 + \frac{1600}{h}$

(iii) Range of h is

(a) (3, 5) (b) $(0, \infty)$ (c) (0, 8) (d) (0, 3)

(iv) Value of h at which $c(h)$ is minimum, is

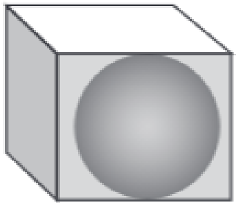
(a) 4 (b) 5 (c) 6 (d) 6.7

(v) The cost of least expensive tank is

(a) ₹ 1020 (b) ₹ 1100 (c) ₹ 1120 (d) ₹ 1220

Case Study 22

Shreya got a rectangular parallelopiped shaped box and spherical ball inside it as return gift. Sides of the box are x, $2x$, and $x/3$, while radius of the ball is r.



Based on the above information, answer the following questions.

(i) If S represents the sum of volume of parallelopiped and sphere, then S can be written as

(a) $\frac{4x^3}{3}+\frac{2}{2}\pi r^2$ (b) $\frac{2x^2}{3}+\frac{4}{3}\pi r^2$ (c) $\frac{2x^3}{3}+\frac{4}{3}\pi r^3$ (d) $\frac{2}{3}x+\frac{4}{3}\pi r$

(ii) If sum of the surface areas of box and ball are given to be constant k^2, then x is equal to

(a) $\sqrt{\frac{k^2-4\pi r^2}{6}}$ (b) $\sqrt{\frac{k^2-4\pi r}{6}}$ (c) $\sqrt{\frac{k^2-4\pi}{6}}$ (d) none of these

(iii) The radius of the ball, when S is minimum, is

(a) $\sqrt{\frac{k^2}{54+\pi}}$ (b) $\sqrt{\frac{k^2}{54+4\pi}}$ (c) $\sqrt{\frac{k^2}{64+3\pi}}$ (d) $\sqrt{\frac{k^2}{4\pi+3}}$

(iv) Relation between length of the box and radius of the ball can be represented as

(a) $x = 2r$ (b) $x = \frac{r}{2}$ (c) $x = \frac{r}{2}$ (d) $x = 3r$

(v) Minimum value of S is

(a) $\frac{k^2}{2(3\pi+54)^{2/3}}$ (b) $\frac{k}{(3\pi+54)^{3/2}}$ (c) $\frac{k^3}{3(4\pi+54)^{1/2}}$ (d) none of these

Case Study 23

A real estate company is going to build a new residential complex. The land they have purchased can hold at most 4500 apartments. Also, if they make x apartments, then the monthly maintenance cost for the whole complex would be as follows : Fixed cost = ₹ 50,00,000. Variable cost = ₹$(160x - 0.04x^2)$

Based on the above information, answer the following questions.

(i) The maintenance cost as a function of x will be
(a) $160x - 0.04x^2$ (b) 5000000
(c) $5000000 + 160x - 0.04x^2$ (d) None of these

(ii) If $C(x)$ denote the maintenance cost function, then maximum value of $C(x)$ occur at x =
(a) 0 (b) 2000 (c) 4500 (d) 5000

(iii) The maximum value of $C(x)$ would be
(a) ₹ 5225000 (b) ₹ 5160000 (c) ₹ 5000000 (d) ₹ 4000000

(iv) The number of apartments, that the complex should have in order to minimize the maintenance cost, is
(a) 4500 (b) 5000 (c) 1750 (d) 3500

(v) If the minimum maintenance cost is attain, then the maintenance cost for each apartment would be
(a) ₹ 1091.11 (b) ₹ 1200 (c) ₹ 1000 (d) ₹ 2000

Case Study 24

Kyra has a rectangular painting canvas having a total area of 24 ft^2 which includes a border of 0.5 ft on the left, right and a border of 0.75 ft on the bottom, top inside it.

Based on the above information, answer the following questions.

(i) If Kyra wants to paint in the maximum area, then she needs to maximize

(a) Area of outer rectangle (b) Area of inner rectangle
(c) Area of top border (d) None of these

(ii) If x is the length of the outer rectangle, then area of inner rectangle in terms of x is

(a) $(x+3)\left(\frac{24}{x}-2\right)$ (b) $(x-1)\left(\frac{24}{x}+1.5\right)$ (c) $(x-1)\left(\frac{24}{x}-1.5\right)$ (d) $(x-1)\left(\frac{24}{x}\right)$

(iii) Find the range of x.

(a) $(1, \infty)$ (b) $(1, 16)$ (c) $(-\infty, 16)$ (d) $(-1, 16)$

(iv) If area of inner rectangle is maximum, then x is equal to

(a) 2 ft (b) 3 ft (c) 4 ft (d) 5 ft

(v) If area of inner rectangle is maximum, then length and breadth of this rectangle are respectively

(a) 3 ft, 4.5 ft (b) 4.5 ft, 5 ft (c) 1 ft, 2 ft (d) 2 ft, 4 ft

Case Study 25

A magazine company in a town has 5000 subscribers on its list and collects fix charges of ₹ 3000 per year from each subscriber. The company proposes to increase the annual charges and it is believed that for every increase of ₹ 1, one subscriber will discontinue service.

Based on the above information, answer the following questions.

(i) If x denote the amount of increase in annual charges, then revenue R, as a function of x can be represented as

(a) $R(x) = 3000 \times 5000 \times x$ (b) $R(x) = (3000 - 2x)(5000 + 2x)$
(c) $R(x) = (5000 + x)(3000 - x)$ (d) $R(x) = (3000 + x)(5000 - x)$

(ii) If magazine company increases ₹ 500 as annual charges, then R is equal to

(a) ₹ 15750000 (b) ₹ 16750000 (c) ₹ 17500000 (d) ₹ 15000000

(iii) If revenue collected by the magazine company is ₹ 15640000, then value of amount increased as annual charges for each subscriber, is

(a) 400 (b) 1600 (c) Both (a) and (b) (d) None of these

(iv) What amount of increase in annual charges will bring maximum revenue?

(a) ₹ 1000 (b) ₹ 2000 (c) ₹ 3000 (d) ₹ 4000

(v) Maximum revenue is equal to

(a) ₹ 15000000 (b) ₹ 16000000 (c) ₹ 20500000 (d) ₹ 25000000

Case Study 26

In a street two lamp posts are 600 feet apart. The light intensity at a distance d from the first (stronger) lamp post is $\frac{1000}{d^2}$, the light intensity at distance d from the second (weaker) lamp post is $\frac{125}{d^2}$ (in both cases the light intensity is inversely proportional to the square of the distance to the light source).

The combined light intensity is the sum of the two light intensities coming from both lamp posts.

Based on the above information, answer the following questions.

(i) If you are in between the lamp posts, at distance x feet from the stronger light, then the formula for the combined light intensity coming from both lamp posts as function of x, is

(a) $\frac{1000}{x^2}+\frac{125}{x^2}$ (b) $\frac{1000}{(600-x^2)}+\frac{125}{x^2}$ (c) $\frac{1000}{x^2}+\frac{125}{(600-x)^2}$ (d) None of these

(ii) The maximum value of x can not be

(a) 100 (b) 200 (c) 600 (d) None of these

(iii) The minimum value of x can not be

(a) 0 (b) 100 (c) 200 (d) None of these

(iv) If $I(x)$ denote the combined light intensity, then $I(x)$ will be minimum when $x =$

(a) 200 (b) 400 (c) 600 (d) 800

(v) The darkest spot between the two lights is

(a) at a distance of 200 feet from the weaker lamp post.

(b) at distance of 200 feet from the stronger lamp post.

(c) at a distance of 400 feet from the weaker lamp post.

(d) None of these

Case Study 27

An open water tank of aluminium sheet of negligible thickness, with a square base and vertical sides, is to be constructed in a farm for irrigation. It should hold 32000 l of water, that comes out from a tube well.

Based on above information, answer the following questions.

(i) If the length, width and height of the open tank be x, x and y m respectively, then total surface area of tank is

(a) $(x^2 + 2xy)$ m^2 (b) $(2x^2 + 4xy)$ m^2 (c) $(2x^2 + 2xy)$ m^2 (d) $(2x^2 + 8xy)$ m^2

(ii) The relation between x and y is

(a) $x^2y = 32$ (b) $xy^2 = 32$ (c) $x^2y^2 = 32$ (d) $xy = 32$

(iii) The outer surface area of tank will be minimum when depth of tank is equal to

(a) half of its width (b) its width (c) $\left(\frac{1}{4}\right)^{th}$ of its width (d) $\left(\frac{1}{3}\right)^{rd}$ of its width

(iv) The cost of material will be least when width of tank is equal to

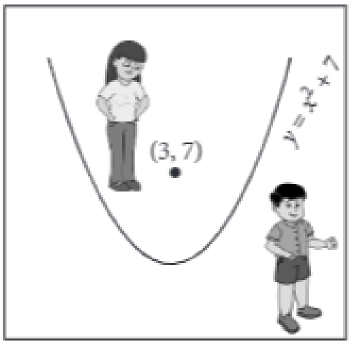
(a) half of its depth (b) twice of its depth (c) $\left(\frac{1}{4}\right)^{th}$ of its depth (d) thrice of its depth

(v) If cost of aluminium sheet is ₹ 360/m^2, then the minimum cost for the construction of tank will be

(a) ₹ 15,000 (b) ₹ 16280 (c) ₹ 17280 (d) ₹ 18280

Case Study 28

A student Arun is running on a playground along the curve given by $y = x^2 + 7$. Another student Manita standing at point (3, 7) on playground wants to hit Arun by paper ball when Arun is nearest to Manita.



Based on above information, answer the following questions.

(i) Arun's position at any value of x will be

(a) $(x^2, y - 7)$ (b) $(x^2, y + 7)$ (c) $(x, x^2 + 7)$ (d) $(x^2, x - 7)$

(ii) Distance (say D) between Arun and Manita will be

(a) $(x - 1)(2x^2 + 2x + 3)$ (b) $(x - 3)^2 + x^4$

(c) $\sqrt{(x-3)^2 + x^4}$ (d) $\sqrt{(x-1)(2x^2 + 2x + 3)}$

(iii) For which real value(s) of x, first derivative of D^2 w.r.t. 'x' will Vanish?

(a) 1 (b) 2 (c) 3 (d) 4

(iv) Find the position of Arun when Manita will hit the paper hall.

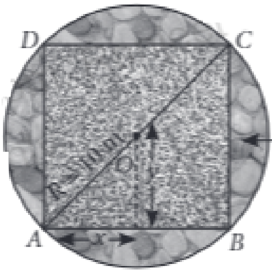
(a) (5, 32) (b) (1, 8) (c) (3, 7) (d) (3, 16)

(v) The minimum value of D is

(a) 3 (b) $\sqrt{3}$ (c) 5 (d) $\sqrt{5}$

Case Study 29

In a society there is a garden in the shape of rectangle inscribed in a circle of radius 10 m as shown in given figure.



Based on the above information, answer the following questions.

(i) If $2x$ and $2y$ denotes the length and breadth in metres, of the rectangular part, then the relation between the variables is

(a) $x^2 - y^2 = 10$ (b) $x^2 + y^2 = 10$ (c) $x^2 + y^2 = 100$ (d) $x^2 - y^2 = 100$

(ii) The area (A) of green grass, in terms of x, is given by

(a) $2x\sqrt{100 - x^2}$ (b) $4x\sqrt{100 - x^2}$ (c) $2x\sqrt{100 + x^2}$ (d) $4x\sqrt{100 + x^2}$

(iii) The maximum value of A is

(a) 100 m^2 (b) 200 m^2 (c) 400 m^2 (d) 1600 m^2

(iv) The value of length of rectangle, when A is maximum, is

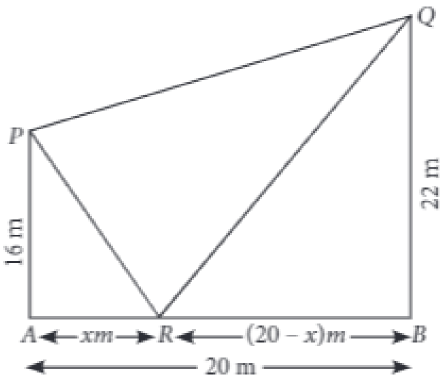
(a) $10\sqrt{2}$ m (b) $20\sqrt{2}$ m (c) 20 m (d) $5\sqrt{2}$ m

(v) The area of gravelling path is

(a) $100 (\pi + 2)$ m^2 (b) $100 (\pi - 2)$ m^2 (c) $200 (\pi + 2)$ m^2 (d) $200 (\pi - 2)$ m^2

Case Study 30

Two multi-storey buildings (represented by AP and BQ) are on opposite side of a 20 m wide road at point A and B respectively. There is a point R on road as shown in figure.



Based on the above information, answer the following questions.

(i) Area of trapezium $ABQP$ is

(a) 380 sq. m (b) 280 sq. m (c) 320 sq. m (d) 430 sq. m

(ii) The length PQ is

(a) 20.5 m (b) 19.80 m (c) 20.88 m (d) 21 m

(iii) Let there be a quantity S such that $S = RP^2 + RQ^2$, then S is given by

(a) $2x^2 - 40x - 1140$ (b) $2x^2 + 40x + 1140$ (c) $2x^2 - 40x + 1140$ (d) $2x^2 + 40x - 1140$

(iv) Find the value of x for which value of S is minimum.

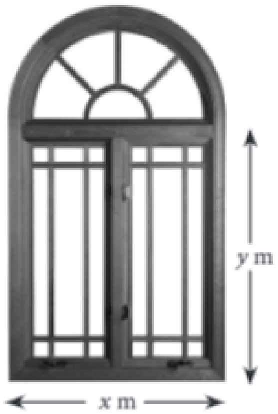
(a) 10 (b) 0 (c) 4 (d) –10

(v) For minimum value of S, find the value of PR and RQ.

(a) 18.50 m, 19.36 m (b) 18.86 m, 24.17 m (c) 17.56 m, 23.29 m (d) None of these

Case Study 31

Rohan, a student of class XII, visited his uncle's flat with his father. He observe that the window of the house is in the form of a rectangle surmounted by a semicircular opening having perimeter 10 m as shown in the figure.



Based on the above information, answer the following questions.

(i) If x and y represents the length and breadth of the rectangular region, then relation between x and y can be represented as

(a) $x + y + \frac{\pi}{2} = 10$ (b) $x + 2y + \frac{\pi x}{2} = 10$ (c) $2x + 2y = 10$ (d) $x + 2y + \frac{\pi}{2} = 10$

(ii) The area (A) of the window can be given by

(a) $A = x - \frac{x^3}{8} - \frac{x^2}{2}$ (b) $A = 5x - \frac{x^2}{2} - \frac{\pi x^2}{8}$

(c) $A = x + \frac{\pi x^3}{8} - \frac{3x^2}{8}$ (d) $A = 5x + \frac{x^2}{2} + \frac{\pi x^2}{8}$

(iii) Rohan is interested in maximizing the area of the whole window, for this to happen, the value of x should be

(a) $\frac{10}{2-\pi}$ (b) $\frac{20}{4-\pi}$ (c) $\frac{20}{4+\pi}$ (d) $\frac{10}{2+\pi}$

(iv) Maximum area of the window is

(a) $\frac{30}{4-\pi}$ (b) $\frac{30}{4+\pi}$ (c) $\frac{50}{4-\pi}$ (d) $\frac{50}{4+\pi}$

(v) For maximum value of A, the breadth of rectangular part of the window is

(a) $\frac{10}{4+\pi}$ (b) $\frac{10}{4-\pi}$ (c) $\frac{20}{4+\pi}$ (d) $\frac{20}{4-\pi}$

Case Study 32

An architecture design a auditorium for a school for its cultural activities. The floor of the auditorium is rectangular in shape and has a fixed perimeter P.

Based on the above information, answer the following questions.

(i) If x and y represents the length and breadth of the rectangular region, then relation between the variable is

(a) $x + y = P$ (b) $x^2 + y^2 = P^2$ (c) $2(x + y) = P$ (d) $x + 2y = P$

(ii) The area (A) of the rectangular region, as a function of x, can be expressed as

(a) $A = Px + \frac{x}{2}$ (b) $A = \frac{Px + x^2}{2}$ (c) $A = \frac{Px - 2x^2}{2}$ (d) $A = \frac{x^2}{2} + Px^2$

(iii) School's manager is interested in maximising the area of floor 'A' for this to be happen, the value of x should be

(a) P (b) $\frac{P}{2}$ (c) $\frac{P}{3}$ (d) $\frac{P}{4}$

(iv) The value of y, for which the area of floor is maximum is

(a) $\frac{P}{2}$ (b) $\frac{P}{3}$ (c) $\frac{P}{4}$ (d) $\frac{P}{16}$

(v) Maximum area of floor is

(a) $\frac{P^2}{16}$ (b) $\frac{P^2}{64}$ (c) $\frac{P^2}{4}$ (d) $\frac{P^2}{28}$

HINTS & EXPLANATIONS

13. **(i) (b):** Since, side of square is of length 20 cm, therefore $x \in (0, 10)$.

(ii) (a) : Clearly, height of open box = x cm

Length of open box = $20 - 2x$

and width of open box = $20 - 2x$

$\therefore$ Volume (V) of the open box

$= x \times (20 - 2x) \times (20 - 2x)$

(iii) (d) : We have, $V = x(20 - 2x)^2$

$\therefore \frac{dV}{dx} = x \cdot 2(20 - 2x)(-2) + (20 - 2x)^2$

$= (20 - 2x)(-4x + 20 - 2x) = (20 - 2x)(20 - 6x)$

Now, $\frac{dV}{dx} = 0 \Rightarrow 20 - 2x = 0$ or $20 - 6x = 0$

$\Rightarrow x = 10$ or $\frac{10}{3}$

(iv) (c) : We have, $V = x(20 - 2x)^2$

and $\frac{dV}{dx} = (20 - 2x)(20 - 6x)$

$\Rightarrow \frac{d^2V}{dx^2} = (20 - 2x)(-6) + (20 - 6x)(-2)$

$= (-2)[60 - 6x + 20 - 6x] = (-2)[80 - 12x] = 24x - 160$

For $x = \frac{10}{3}, \frac{d^2V}{dx^2} < 0$

and for $x = 10, \frac{d^2V}{dx^2} > 0$

So, volume will be maximum when $x = \frac{10}{3}$.

(v) (d) : We have, $V = x(20 - 2x)^2$, which will be maximum when $x = \frac{10}{3}$.

$\therefore$ Maximum volume $= \frac{10}{3}\left(20 - 2\times\frac{10}{3}\right)^2$

$= \frac{10}{3}\times\frac{40}{3}\times\frac{40}{3} = \frac{16000}{27}\ \text{cm}^3$

14. (i) (b): To create a garden using 200 ft fencing, we need to maximise its area.

(ii) (c) : Required relation is given by $2x + y = 200$.

(iii) (c) : Area of garden as a function of x can be represented as

$A(x) = x \cdot y = x(200 - 2x) = 200x - 2x^2$

(iv) (a) : $A(x) = 200x - 2x^2 \Rightarrow A'(x) = 200 - 4x$

For the area to be maximum $A'(x) = 0$

$\Rightarrow 200 - 4x = 0 \Rightarrow x = 50$ ft.

(v) (c) : Maximum area of the garden

$= 200(50) - 2(50)^2 = 10000 - 5000 = 5000$ sq. ft

15. (i) (a): Let x be the number of extra days after 1st July.

$\therefore$ Price = ₹$(300 - 3 \times x)$ = ₹$(300 - 3x)$

Quantity = 80 quintals + x(1 quintal per day)

$= (80 + x)$ quintals

(ii) (b) : $R(x)$ = Quantity × Price

$= (80 + x)(300 - 3x) = 24000 - 240x + 300x - 3x^2$

$= 24000 + 60x - 3x^2$

(iii) (a) : We have, $R(x) = 24000 + 60x - 3x^2$

$\Rightarrow R'(x) = 60 - 6x \Rightarrow R''(x) = -6$

For $R(x)$ to be maximum, $R'(x) = 0$ and $R''(x) < 0$

$\Rightarrow 60 - 6x = 0 \Rightarrow x = 10$

(iv) (a) : Shyam's father will attain maximum revenue after 10 days.

So, he should harvest the onions after 10 days of 1st July *i.e.*, on 11th July.

(v) (c) : Maximum revenue is collected by Shyam's father when $x = 10$

$\therefore$ Maximum revenue $= R(10)$

$= 24000 + 60(10) - 3(10)^2 = 24000 + 600 - 300 = 24300$

16. (i) (a): Let x be the charges per bike per day and n be the number of bikes rented per day.

$R(x) = n \times x = (2000 - 10x)\,x = -10x^2 + 2000x$

(ii) (b) : We have, $R(x) = 2000x - 10x^2$

$\Rightarrow R'(x) = 2000 - 20x$

For $R(x)$ to be maximum or minimum, $R'(x) = 0$

$\Rightarrow 2000 - 20x = 0 \Rightarrow x = 100$

Also, $R''(x) = -20 < 0$

Thus, $R(x)$ is maximum at $x = 100$

(iii) (c) : If company charge ₹ 200 or more, they will not rent any bike. Therefore, revenue collected by him will be zero.

(iv) (c) : If $x = 105$, number of bikes rented per day is given by

$n = 2000 - 10 \times 105 = 950$

(v) (d) : At $x = 100$, $R(x)$ is maximum.

$\therefore$ Maximum revenue $= R(100)$

$= -10(100)^2 + 2000(100) =$ ₹ 1,00,000

17. (i) (c): If P is the rent price per apartment and N is the number of rented apartment, the profit is given by

$NP - 500\,N = N(P - 500)$

[$\because$ ₹ 500/month is the maintenance charges for each occupied unit]

(ii) (c) : Now, if x be the number of non-rented apartments, then $N = 50 - x$ and $P = 10000 + 250\,x$

Thus, profit $= N(P - 500) = (50 - x)(10000 + 250\,x - 500)$

$= (50 - x)(9500 + 250\,x) = 250(50 - x)(38 + x)$

(iii) (b) : Clearly, if $P = 10500$, then

$10500 = 10000 + 250\,x \Rightarrow x = 2 \Rightarrow N = 48$

(iv) (a) : Also, if $P = 11000$, then

$11000 = 10000 + 250\,x \Rightarrow x = 4$ and so profit

$P(4) = 250(50 - 4)(38 + 4) =$ ₹ 4,83,000

(v) (b) : We have, $P(x) = 250(50 - x)(38 + x)$

Now, $P'(x) = 250[50 - x - (38 + x)] = 250[12 - 2x]$

For maxima/minima, put $P'(x) = 0$

$\Rightarrow 12 - 2x = 0 \Rightarrow x = 6$

Thus, price per apartment is, $P = 10000 + 1500 = 11500$

Hence, the rent that maximizes the profit is ₹ 11500.

18. (i) (a): Let p be the price per ticket and x be the number of tickets sold.

Then, revenue function $R(x) = p \times x = \left(15 - \frac{x}{3000}\right)x$

$= 15x - \frac{x^2}{3000}$

(ii) (c) : Since, more than 36000 tickets cannot be sold. So, range of x is [0, 36000].

(iii) (c) : We have, $R(x) = 15x - \frac{x^2}{3000}$

$\Rightarrow R'(x) = 15 - \frac{x}{1500}$

For maxima/minima, put $R'(x) = 0$

$$\Rightarrow \quad 15 - \frac{x}{1500} = 0 \Rightarrow x = 22500$$

Also, $R''(x) = -\frac{1}{1500} < 0$.

(iv) (d) : Maximum revenue will be at $x = 22500$

$\therefore$ Price of a ticket $= 15 - \frac{22500}{3000} = 15 - 7.5 = ₹\,7.5$

(v) (d) : Number of spectators will be equal to number of tickets sold.

$\therefore$ Required number of spectators = 22500

19. (i) (d): Given, r cm is the radius and h cm is the height of required cylindrical can.

Given that, volume = 3 l = 3000 cm^3 $(\because 1\,l = 1000 \text{ cm}^3)$

$$\Rightarrow \quad \pi r^2 h = 3000 \Rightarrow h = \frac{3000}{\pi r^2}$$

Now, the surface area, as a function of r is given by

$$S(r) = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r\left(\frac{3000}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{6000}{r}$$

(ii) (c) : Now, $S(r) = 2\pi r^2 + \frac{6000}{r}$

$$\Rightarrow \quad S'(r) = 4\pi r - \frac{6000}{r^2}$$

To find critical points, put $S'(r) = 0$

$$\Rightarrow \quad \frac{4\pi r^3 - 6000}{r^2} = 0$$

$$\Rightarrow \quad r^3 = \frac{6000}{4\pi} \Rightarrow r = \left(\frac{1500}{\pi}\right)^{1/3}$$

Also, $S''(r)\Big|_{r=\sqrt[3]{\frac{1500}{\pi}}} = 4\pi + \frac{12000 \times \pi}{1500}$

$= 4\pi + 8\pi = 12\pi > 0$

Thus, the critical point is the point of minima.

(iii) (b) : The cost of material for the tin can is minimized when $r = \sqrt[3]{\frac{1500}{\pi}}$ cm and the height is

$$\frac{3000}{\pi\left(\sqrt[3]{\frac{1500}{\pi}}\right)^2} = 2\sqrt[3]{\frac{1500}{\pi}} \text{ cm.}$$

(iv) (a) : We have, minimum surface area $= \frac{2\pi r^3 + 6000}{r}$

$$= \frac{2\pi \cdot \frac{1500}{\pi} + 6000}{\sqrt[3]{\frac{1500}{\pi}}} = \frac{9000}{7.8} = 1153.84 \text{ cm}^2$$

Cost of 1 m^2 material = ₹100

$\therefore$ Cost of 1 cm^2 material = ₹ $\frac{1}{100}$

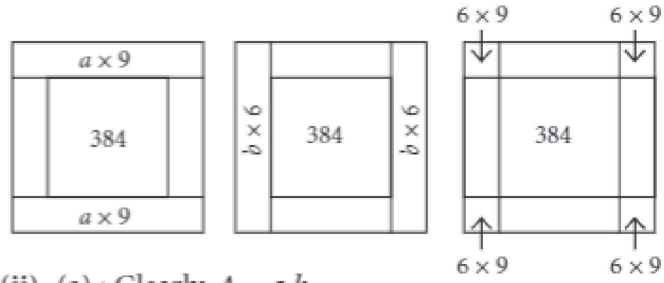
$\therefore$ Minimum cost = ₹ $\frac{1153.84}{100}$ = ₹ 11.538

(v) (c) : To minimize the cost we need to minimize the total surface area.

20. (i) (a): Let A be the area of the poster, then

$A = 864 + 2(a \cdot 9) + 2(b \cdot 6) - 4(6 \cdot 9)$

$= 864 + 18a + 12b - 216 = 648 + 18a + 12b$



(ii) (a) : Clearly, $A = a \cdot b$

$\therefore$ $648 + 18a + 12b = ab$

$\Rightarrow$ $ab - 18a = 648 + 12b \Rightarrow a(b - 18) = 648 + 12b$

$$\Rightarrow \quad a = \frac{648 + 12b}{b - 18}$$

(iii) (b) : Since, $A = a \cdot b$, therefore

$$A = \left(\frac{648 + 12b}{b - 18}\right) \cdot b = \frac{648b + 12b^2}{b - 18} \quad \left[\because a = \frac{648 + 12b}{b - 18}\right]$$

(iv) (a) : Clearly,

$$A'(b) = \frac{(b-18)(648 + 24b) - (648b + 12b^2)}{(b-18)^2}$$

$$= \frac{12[b^2 - 36b - 972]}{(b-18)^2}$$

For minimum, consider $A'(b) = 0$

$\Rightarrow$ $b^2 - 36b - 972 = 0$

$\Rightarrow$ $b^2 - 54b + 18b - 972 = 0$

$\Rightarrow$ $b(b - 54) + 18(b - 54) = 0$

$\Rightarrow$ $b = -18$ or $b = 54$

$\because$ b is height, therefore can't be negative.

So, $b = 54$.

(v) (b) : Since, $a = \frac{648 + 12b}{b - 18}$

$$\therefore \quad a = \frac{648 + 12 \times 54}{54 - 18} = \frac{648 + 648}{36} = 36$$

21. (i) (d): In order to make least expensive water tank, Nitin need to minimize its cost.

(ii) (d) : Let l ft be the length and h ft be the height of the tank. Since breadth is equal to 5 ft. (Given)

$\therefore$ Two sides will be $5h$ sq. feet and two sides will be lh sq. feet. So, the total area of the sides is $(10h + 2lh)$ft^2

Cost of the sides is ₹ 10 per sq. foot. So, the cost to build the sides is $(10h + 2lh) \times 10 =$ ₹ $(100h + 20lh)$

Also, cost of base $= (5\,l) \times 20 =$ ₹ $100\,l$

$\therefore$ Total cost of the tank in ₹ is given by

$c = 100\,h + 20\,l\,h + 100\,l$

Since, volume of tank = 80 ft^3

$\therefore\ 5\,l\,h = 80 \text{ ft}^3 \ \therefore\ l = \frac{80}{5h} = \frac{16}{h}$

$\therefore\ c(h) = 100\,h + 20\left(\frac{16}{h}\right)h + 100\left(\frac{16}{h}\right)$

$= 100\,h + 320 + \frac{1600}{h}$

(iii) (b) : Since, all side lengths must be positive.

$\therefore\ h > 0$ and $\frac{16}{h} > 0$

Since, $\frac{16}{h} > 0$, whenever $h > 0$

$\therefore$ Range of h is $(0, \infty)$.

(iv) (a) : To minimize cost, $\frac{dc}{dh} = 0$

$\Rightarrow\ 100 - \frac{1600}{h^2} = 0$

$\Rightarrow\ 100\,h^2 = 1600 \Rightarrow h^2 = 16 \Rightarrow h = \pm 4$

$\Rightarrow\ h = 4$ [$\because$ height can not be negative]

(v) (c) : Cost of least expensive tank is given by

$c(4) = 400 + 320 + \frac{1600}{4}$

$= 720 + 400 =$ ₹ 1120

22. (i) (c) : Let S be the sum of volume of parallelopiped and sphere, then

$$S = x(2x)\left(\frac{x}{3}\right) + \frac{4}{3}\pi r^3 = \frac{2x^3}{3} + \frac{4}{3}\pi r^3 \quad \text{... (1)}$$

(ii) (a) : Since, sum of surface area of box and sphere is given to be constant k^2.

$$\therefore\ 2\left(x \times 2x + 2x \times \frac{x}{3} + \frac{x}{3} \times x\right) + 4\pi r^2 = k^2$$

$\Rightarrow\ 6x^2 + 4\pi r^2 = k^2$

$$\Rightarrow\ x^2 = \frac{k^2 - 4\pi r^2}{6} \Rightarrow x = \sqrt{\frac{k^2 - 4\pi r^2}{6}} \quad \text{... (2)}$$

(iii) (b) : From (1) and (2), we get

$$S = \frac{2}{3}\left(\frac{k^2 - 4\pi r^2}{6}\right)^{3/2} + \frac{4}{3}\pi r^3$$

$$= \frac{2}{3 \times 6\sqrt{6}}(k^2 - 4\pi r^2)^{3/2} + \frac{4}{3}\pi r^3$$

$$\Rightarrow\ \frac{dS}{dr} = \frac{1}{9\sqrt{6}}\frac{3}{2}(k^2 - 4\pi r^2)^{1/2}(-8\pi r) + 4\pi r^2$$

$$= 4\pi r\left[r - \frac{1}{3\sqrt{6}}\sqrt{k^2 - 4\pi r^2}\right]$$

For maximum/minimum, $\frac{dS}{dr} = 0$

$$\Rightarrow\ \frac{-4\pi r}{3\sqrt{6}}\sqrt{k^2 - 4\pi r^2} = -4\pi r^2$$

$\Rightarrow\ k^2 - 4\pi r^2 = 54r^2$

$$\Rightarrow\ r^2 = \frac{k^2}{54 + 4\pi} \Rightarrow r = \sqrt{\frac{k^2}{54 + 4\pi}} \quad \text{... (3)}$$

(iv) (d) : Since, $x^2 = \frac{k^2 - 4\pi r^2}{6} = \frac{1}{6}\left[k^2 - 4\pi\left(\frac{k^2}{54 + 4\pi}\right)\right]$

[From (2) and (3)]

$$= \frac{9k^2}{54 + 4\pi} = 9\left(\frac{k^2}{54 + 4\pi}\right) = 9r^2 = (3r)^2$$

$\Rightarrow\ x = 3r$

(v) (c) : Minimum value of S is given by

$$\frac{2}{3}(3r)^3 + \frac{4}{3}\pi r^3$$

$$= 18r^3 + \frac{4}{3}\pi r^3 = \left(18 + \frac{4}{3}\pi\right)r^3$$

$$= \left(18 + \frac{4}{3}\pi\right)\left(\frac{k^2}{54 + 4\pi}\right)^{3/2} \quad \text{[Using (3)]}$$

$$= \frac{1}{3}\frac{k^3}{(54 + 4\pi)^{1/2}}$$

23. (i) (c) : Let $C(x)$ be the maintenance cost function, then $C(x) = 5000000 + 160x - 0.04x^2$

(ii) (b) : We have, $C(x) = 5000000 + 160x - 0.04x^2$

Now, $C'(x) = 160 - 0.08\,x$

For maxima/minima, put $C'(x) = 0$

$\Rightarrow\ 160 = 0.08x$

$\Rightarrow\ x = 2000$

(iii) (b) : Clearly, from the given condition we can see that we only want critical points that are in the interval [0, 4500]

Now, we have $C(0) = 5000000$

$C(2000) = 5160000$

and $C(4500) = 4910000$

$\therefore$ Maximum value of $C(x)$ would be ₹ 5160000

(iv) (a) : The complex must have 4500 apartments to minimise the maintenance cost.

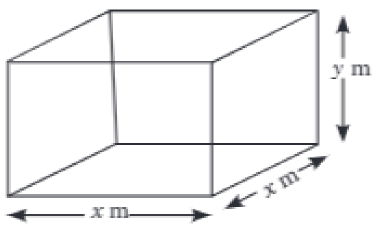
(v) (a) : The minimum maintenance cost for each apartment woud be ₹ 1091.11

24. (i) (b): In order to paint in the maximum area, Kyra needs to maximize the area of inner rectangle.
(ii) (c) : Let x be the length and y be the breadth of outer rectangle.
$\therefore$ Length of inner rectangle = $x - 1$
and breadth of inner rectangle = $y - 1.5$
$\therefore$ $A(x) = (x - 1)(y - 1.5)$ [$\because xy = 24$ (given)]
$= (x-1)\left(\frac{24}{x}-1.5\right)$
(iii) (b) : Dimensions of rectangle (outer/inner) should be positive.
$\therefore$ $x - 1 > 0$ and $\frac{24}{x}-1.5>0$
$\Rightarrow$ $x > 1$ and $x < 16$
$\therefore$ Range of x is (1, 16).
(iv) (c) : We have, $A(x) = (x-1)\left(\frac{24}{x}-1.5\right)$
$\Rightarrow$ $A'(x) = (x-1)\left(\frac{-24}{x^2}\right)+\left(\frac{24}{x}-1.5\right)=\frac{24}{x^2}-1.5$
and $A''(x) = \frac{-48}{x^3}$
For $A(x)$ to be maximum or minimum, $A'(x) = 0$
$\Rightarrow$ $-1.5+\frac{24}{x^2}=0 \Rightarrow x^2=16 \Rightarrow x=\pm 4$
$\therefore$ $x = 4$ [Since, length can't be negative]
Also, $A''(4) = \frac{-48}{4^3} < 0$
Thus, at $x = 4$, area is maximum.
(v) (a) : If area of inner rectangle is maximum, then
Length of inner rectangle = $x - 1 = 4 - 1 = 3$ ft
And breadth of inner rectangle = $y - 1.5 = \frac{24}{x} - 1.5$
$= \frac{24}{4} - 1.5 = 6 - 1.5 = 4.5$ ft

25. (i) (d): If x be the amount of increase in annual charges, then number of subscriber reduces to $5000 - x$
$\therefore$ Revenue, $R(x) = (3000 + x)(5000 - x)$
$= 15000000 + 2000x - x^2, 0 < x < 5000$
(ii) (a) : Clearly, at $x = 500$
$R(500) = 15000000 + 2000(500) - (500)^2$
$= 15000000 + 1000000 - 250000 =$ ₹ 15750000
(iii) (c) : Since, $15000000 + 2000x - x^2 = 15640000$ (Given)
$\Rightarrow$ $x^2 - 2000x + 640000 = 0$
$\Rightarrow$ $x^2 - 1600x - 400x + 640000 = 0$
$\Rightarrow$ $x(x - 1600) - 400(x - 1600) = 0 \Rightarrow x = 400, 1600$
(iv) (a) : $\frac{dR}{dx} = 2000 - 2x$ and $\frac{d^2R}{dx^2} = -2 < 0$
For maximum revenue, $\frac{dR}{dx} = 0 \Rightarrow x = 1000$
$\therefore$ Required amount = ₹ 1000
(v) (b) : Maximum revenue = $R(1000)$
$= (3000 + 1000)(5000 - 1000)$
$= 4000 \times 4000 =$ ₹ 16000000

26. (i) (c): Since, the distance is x feet from the stronger light, therefore the distance from the weaker light will be $600 - x$.
So, the combined light intensity from both lamp posts is given by $\frac{1000}{x^2}+\frac{125}{(600-x)^2}$.
(ii) (c) : Since, the person is in between the lamp posts, therefore x will lie in the interval (0, 600).
So, maximum value of x can't be 600.
(iii) (a) : Since, $0 < x < 600$, therefore minimum value of x can't be 0.
(iv) (b) : We have, $I(x)=\frac{1000}{x^2}+\frac{125}{(600-x)^2}$
$\Rightarrow$ $I'(x)=\frac{-2000}{x^3}+\frac{250}{(600-x)^3}$ and
$\Rightarrow$ $I''(x)=\frac{6000}{x^4}+\frac{750}{(600-x)^4}$
For maxima/minima, $I'(x) = 0$
$\Rightarrow$ $\frac{2000}{x^3}=\frac{250}{(600-x)^3} \Rightarrow 8(600-x)^3 = x^3$
Taking cube root on both sides, we get
$2(600 - x) = x \Rightarrow 1200 = 3x \Rightarrow x = 400$
Thus, $I(x)$ is minimum when you are at 400 feet from the strong intensity lamp post.
(v) (a) : Since, $I(x)$ is minimum when $x = 400$ feet, therefore the darkest spot between the two light is at a distance of 400 feet from stronger lamp post, *i.e.*, at a distance of $600 - 400 = 200$ feet from the weaker lamp post.

27. (i) (d): Since the tank is open from the top, therefore the total surface area is
= (Outer + Inner) surface area
$= 2(x \times x + 2(xy + yx)) = 2(x^2 + 2(2xy)) = (2x^2 + 8xy)$ m^2



(ii) **(a) :** Since, volume of tank should be 32000 *l*.

$\therefore \quad x^2y \text{ m}^3 = 32000\, l = 32 \text{ m}^3 \quad [\because 1 \text{ litre} = 0.001 \text{ m}^3]$

So, $x^2y = 32$

(iii) **(a) :** Let S be the outer surface area of tank.

Then, $S = x^2 + 4xy$

$\Rightarrow \quad S(x) = x^2 + 4x \cdot \dfrac{32}{x^2} = x^2 + \dfrac{128}{x} \quad [\because x^2y = 32]$

$\Rightarrow \quad \dfrac{dS}{dx} = 2x - \dfrac{128}{x^2}$ and $\dfrac{d^2S}{dx^2} = 2 + \dfrac{256}{x^3}$

For maximum or minimum values of S, consider

$\dfrac{dS}{dx} = 0$

$\Rightarrow \quad 2x = \dfrac{128}{x^2} \Rightarrow x^3 = 64 \Rightarrow x = 4 \text{ m}$

At $x = 4$, $\dfrac{d^2S}{dx^2} = 2 + \dfrac{256}{4^3} = 2 + 4 = 6 > 0$

$\therefore \quad S$ is minimum when $x = 4$

Now as $x^2y = 32$, therefore $y = 2$

Thus, $x = 2y$

(iv) **(b) :** Since, surface area is minimum when $x = 2y$, therefore cost of material will be least when $x = 2y$. Thus, cost of material will be least when width is equal to twice of its depth.

(v) **(c) :** Since, minimum surface area

$= x^2 + 4xy = 4^2 + 4 \times 4 \times 2 = 48 \text{ m}^2$ and

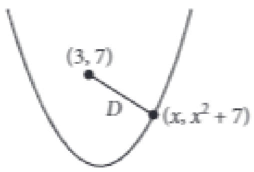
cost per m^2 = ₹ 360

$\therefore \quad$ Minimum cost is = ₹ (48×360) = ₹ 17280

28. **(i)** **(c) :** For all values of x, $y = x^2 + 7$

$\therefore \quad$ Arun's position at any point of x will be $(x, x^2 + 7)$

(ii) **(c) :** Distance between Arun and Manita, *i.e.*, D



$= \sqrt{(x-3)^2 + (x^2 + 7 - 7)^2}$

$= \sqrt{(x-3)^2 + x^4}$

(iii) **(a) :** We have, $D = \sqrt{(x-3)^2 + x^4}$

$\therefore \quad D^2 = (x-3)^2 + x^4$

Now, $\dfrac{d}{dx}(D^2) = 2(x-3) + 4x^3 = 0$

$\Rightarrow \quad 4x^3 + 2x - 6 = 0 \Rightarrow 2x^3 + x - 3 = 0$

$\Rightarrow \quad (x-1)(2x^2 + 2x + 3) = 0$

$\therefore \quad x = 1$

$(\because\ 2x^2 + 2x + 3 = 0$ will give imaginary values)

(iv) **(b) :** We have, $D = \sqrt{(x-3)^2 + x^4}$

$D'(x) = \dfrac{2(x-3) + 4x^3}{2\sqrt{(x-3)^2 + x^4}} = 0$

$\Rightarrow \quad 2x^3 + x - 3 = 0$

$\Rightarrow \quad x = 1$

Clearly, $D''(x)$ at $x = 1$ is > 0

$\therefore \quad$ Value of x for which D will be minimum is 1

For $x = 1$, $y = 8$.

Thus, the required position is (1, 8).

(v) **(d) :** Minimum value of $D = \sqrt{(1-3)^2 + (1)^4}$

$= \sqrt{4+1} = \sqrt{5}$

29. **(i)** **(c) :** Length, $AB = 2x$

Breadth, $BC = 2y$

Also, radius, $OA = 10$

$\therefore \quad AC = 20$

In ΔABC, $AB + BC^2 = AC^2$

$\Rightarrow \quad (2x)^2 + (2y)^2 = (20)^2$

$\Rightarrow \quad x^2 + y^2 = 100$

(ii) **(b) :** Area of green grass = Area of rectangular part

$\therefore \quad A = 2x \cdot 2y$ [$\because$ Area of rectangle = length $\times$ breadth]

$= 4xy = 4x\sqrt{100 - x^2} \quad [\because\ x^2 + y^2 = 100]$

(iii) **(b) :** We have, $A = 4x\sqrt{100 - x^2}$

$\dfrac{dA}{dx} = \dfrac{4x(-2x)}{2\sqrt{100 - x^2}} + \sqrt{100 - x^2} \cdot 4$

$= \dfrac{-4x^2 + 4(100 - x^2)}{\sqrt{100 - x^2}}$

For maximum value, $\dfrac{dA}{dx} = 0$

$\Rightarrow \quad -4x^2 + 400 - 4x^2 = 0$

$\Rightarrow \quad -8x^2 + 400 = 0$

$\Rightarrow \quad x^2 = 50 \Rightarrow x = 5\sqrt{2}$

At $x = 5\sqrt{2}$,

$A = 4x\sqrt{100 - x^2}$

$= 4 \times 5\sqrt{2} \cdot \sqrt{100 - 50} = 4 \times 5\sqrt{2} \times 5\sqrt{2} = 200 \text{ m}^2$

(iv) **(a) :** Length of rectangle for which A is maximum

$= 2 \times 5\sqrt{2} = 10\sqrt{2}$

(v) **(b) :** Area of gravelling path $= \pi(10)^2 - 200$

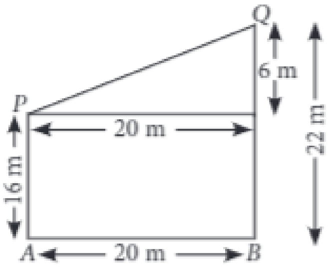
$= 100(\pi - 2) \text{ m}^2$

30. **(i)** **(a) :** Area of trapezium $= \dfrac{1}{2} \times$ (sum of parallel sides) $\times$ distance between parallel sides

$= \dfrac{1}{2} \times (16 + 22) \times 20 = 380 \text{ m}^2$

(ii) **(c) :** $PQ^2 = 6^2 + (20)^2 = 36 + 400 = 436$

$\therefore \quad PQ = \sqrt{436} = 20.88 \text{ m}$



(iii) (c) : We have, $S = RP^2 + RQ^2$ ...(i)

Since, $RP^2 = (16)^2 + x^2 = 256 + x^2$

and $RQ^2 = (22)^2 + (20 - x)^2 = 484 + 400 + x^2 - 40x$

$\therefore \quad S = 2x^2 - 40x + 1140$

(iv) (a) : We have, $S(x) = 2x^2 - 40x + 1140$

$\therefore \quad S'(x) = 4x - 40$

For minimum value of x, $S'(x) = 0$

$\Rightarrow \quad 4x - 40 = 0 \Rightarrow x = 10$

Clearly, at $x = 10$, $S''(x) = 4 > 0$

(v) (b) : At $x = 10$, $PR^2 = 16^2 + x^2$

$= (16)^2 + (10)^2 = 256 + 100 = 356$

$\therefore \quad PR = \sqrt{356} = 18.86$ m

Also, $RQ^2 = (22)^2 + (20 - 10)^2 = 484 + 100 = 584$

$\therefore \quad RQ = \sqrt{584} = 24.17$ m

31. (i) (b): Given, perimeter of window = 10 m

$\therefore \quad x + y + y +$ perimeter of semicircle = 10

$\Rightarrow \quad x + 2y + \pi\dfrac{x}{2} = 10$

(ii) (b) : $A = x \cdot y + \dfrac{1}{2}\pi\left(\dfrac{x}{2}\right)^2$

$$= x\left(5 - \frac{x}{2} - \frac{\pi x}{4}\right) + \frac{1}{2}\frac{\pi x^2}{4} \quad \left[\because \text{From (i), } y = 5 - \frac{x}{2} - \frac{\pi x}{4}\right]$$

$$= 5x - \frac{x^2}{2} - \frac{\pi x^2}{4} + \frac{\pi x^2}{8} = 5x - \frac{x^2}{2} - \frac{\pi x^2}{8}$$

(iii) (c) : We have, $A = 5x - \dfrac{x^2}{2} - \dfrac{\pi x^2}{8}$

$$\Rightarrow \quad \frac{dA}{dx} = 5 - x - \frac{\pi x}{4}$$

Now, $\dfrac{dA}{dx} = 0 \Rightarrow 5 = x + \dfrac{\pi x}{4}$

$$\Rightarrow \quad x(4 + \pi) = 20 \Rightarrow x = \frac{20}{4 + \pi}$$

$$\left[\text{Clearly, } \frac{d^2A}{dx^2} < 0 \text{ at } x = \frac{20}{4 + \pi}\right]$$

(iv) (d) : At $x = \dfrac{20}{4 + \pi}$

$$A = 5\left(\frac{20}{4+\pi}\right) - \left(\frac{20}{4+\pi}\right)^2 \frac{1}{2} - \frac{\pi}{8}\left(\frac{20}{4+\pi}\right)^2$$

$$= \frac{100}{4+\pi} - \frac{200}{(4+\pi)^2} - \frac{50\pi}{(4+\pi)^2}$$

$$= \frac{(4+\pi)(100) - 200 - 50\pi}{(4+\pi)^2} = \frac{400 + 100\pi - 200 - 50\pi}{(4+\pi)^2}$$

$$= \frac{200 + 50\pi}{(4+\pi)^2} = \frac{50(4+\pi)}{(4+\pi)^2} = \frac{50}{4+\pi}$$

(v) (a) : We have, $y = 5 - \dfrac{x}{2} - \dfrac{\pi x}{4} = 5 - x\left(\dfrac{1}{2} + \dfrac{\pi}{4}\right)$

$$= 5 - x\left(\frac{2+\pi}{4}\right) = 5 - \left(\frac{20}{4+\pi}\right)\left(\frac{2+\pi}{4}\right)$$

$$= 5 - 5\frac{(2+\pi)}{4+\pi} = \frac{20 + 5\pi - 10 - 5\pi}{4+\pi} = \frac{10}{4+\pi}$$

32. (i) (c) : Perimeter of floor = 2(length + breadth)

$\Rightarrow \quad P = 2(x + y)$

(ii) (c) : Area, A = length × breadth

$\Rightarrow \quad A = xy$

Since, $P = 2(x + y)$

$$\Rightarrow \quad \frac{P - 2x}{2} = y$$

(iii) (d) : We have, $A = \dfrac{1}{2}(Px - 2x^2)$

$$\frac{dA}{dx} = \frac{1}{2}(P - 4x) = 0$$

$$\Rightarrow \quad P - 4x = 0 \Rightarrow x = \frac{P}{4}$$

Clearly, at $x = \dfrac{P}{4}$, $\dfrac{d^2A}{dx^2} = -2 < 0$

$\therefore \quad$ Area is maximum at $x = \dfrac{P}{4}$.

(iv) (c) : We have, $y = \dfrac{P - 2x}{2} = \dfrac{P}{2} - \dfrac{P}{4} = \dfrac{P}{4}$

(v) (a) : We have, $A = xy = \dfrac{P}{4} \cdot \dfrac{P}{4} = \dfrac{P^2}{16}$